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A Finite Difference Technique for Singularly Perturbed Two-Point Boundary value Problem using Deviating Argument

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Abstract— In this paper, we have presented a finite difference technique to solve singularly perturbed two-point boundary value problem using deviating argument. We have replaced the given second order boundary value problem by an asymptotically equivalent first order differential equation with deviating argument. We have applied a fourth order finite difference approximation for the first derivative and obtained a tridiagonal system. Then, we have solved this tridiagonal system efficiently by discrete invariant imbedding algorithm. This method is iterative on the deviating argument. We have presented numerical results of several model examples to support the proposed method.

Index Terms — Singularly perturbed two-point boundary value problem, Boundary layer, Deviating argument, Finite difference, Taylor series, Tridiagonal system.

1 Introduction

Singularly perturbed boundary value problems arise frequently in many areas of science and engineering such as heat transfer problem with large Peclet numbers, Navier–Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, reaction– diffusion process, quantum mechanics, optimal control etc. due to the variation in the width of the layer with respect to the small perturbation parameter ε. Several difficulties are experienced in solving the singular perturbation problems using standard numerical methods with uniform mesh. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution derivatives are extremely large.

These types of problems are discussed asymptotically by Bellman [1], Bender and Orszag [2], Kevorkian and Cole [5], Nayfeh [7], O'Malley [8] and numerically by Van Veldhuizen [10], Miller [6], Kadalbajoo and Reddy [4], Soujanya et. al. [9]. It is well-known that replacing the first derivative by central difference is not suitable, i.e., no resemblance at all exists between the solution of the differential equation and the solution of the difference equation. This difficulty can be removed by approximating the first derivative by fourth order difference.

Hence, in this paper, we have presented a finite difference technique to solve singularly perturbed two-point boundary value problem using deviating argument. We have replaced the given second order boundary value problem by an asymptotically equivalent first order differential equation with deviating argument. We have applied a fourth order finite difference approximation for the first derivative and obtained a tridiagonal system. Then, we have solved this tridiagonal system efficiently by discrete invariant imbedding algorithm. This method is iterative on the deviating argument. We have presented numerical results of several model examples to support the proposed method.

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2 Description of the Method

2.1 Left – end Boundary Layer

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0, 1] \tag{1}$$

with boundary conditions $y(0) = \alpha$ and $y(1) = \beta$ (2)

where ε is a small positive parameter ($0 < \varepsilon << 1$) and α, β are known constants.

We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0, 1]. Furthermore, we assume that $b(x) \le 0$, $a(x) \ge M > 0$ throughout the interval [0, 1], where M is some positive constant. Under these assumptions, (1) – (2) has a unique solution $y(x)$ which in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

First, we replace the original second order differential equation (1) by an asymptotically equivalent first order differential equation with a small deviating argument, and then solving it efficiently by employing finite differences.

Let γ be a small positive deviating argument $0 < \gamma << 1$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y(x-\gamma) = y(x) - \gamma y'(x) + \frac{\gamma^2}{2} y''(x)$$

$$y''(x) = \frac{2y(x-\gamma) - 2y(x) + 2\gamma y'(x)}{\gamma^2} \tag{3}$$

and consequently, equation (1) is replaced by the following first order differential equation with a small deviating argument:

$$y'(x) = p(x)y(x-\gamma) + q(x)y(x) + r(x)\text{, for } \gamma \le x \le 1 \tag{4}$$

where

$$p(x) = \frac{-2\varepsilon}{2\gamma\varepsilon + \gamma^2 a(x)},\quad q(x) = \frac{2\varepsilon - \gamma^2 b(x)}{2\gamma\varepsilon + \gamma^2 a(x)}$$

$$r(x) = \frac{\gamma^2 f(x)}{2\gamma\varepsilon + \gamma^2 a(x)}$$

The transition from equation (1) to equation (4) is admitted, because of the condition that γ is small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in El'sgol'ts and Norkin [3].

We now describe a new finite difference method to solve equation (4). We divide the interval [0, 1] into N equal subintervals of mesh size $h=1/N$ so that $x_i = ih$, $i = 0, 1, 2, \ldots, N$. The equation (4) at $x = x_i$ for i = 1, 2, …., N-1 becomes

$$y'(x_i) = p(x_i)y(x_i - \gamma) + q(x_i)y(x_i) + r(x_i) \tag{5}$$

Taking fourth order finite difference approximation

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h\left(1 + \frac{\delta^2}{6}\right)} + O\left(h^4\right)$$

in the above equation, where $\delta^2 y_i = y_{i-1} - 2y_i + y_{i+1}$, we get

$$\frac{y_{i+1} - y_{i-1}}{2h\left(1 + \frac{\delta^2}{6}\right)} = p(x_i)y(x_i - \delta) + q(x_i)y(x_i) + r(x_i)$$

$$\frac{y_{i+1} - y_{i-1}}{2h} = \left(1 + \frac{\delta^2}{6}\right)\left(p(x_i)y(x_i - \delta) + q(x_i)y(x_i) + r(x_i)\right)$$

After simple manipulation, we get three term recurrence relation

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i\text{, for } i = 1, 2, \ldots, N\text{-1} \tag{6}$$

where

$$E_i = \frac{-1}{2h} - \frac{p_{i-1}}{6}\left(1 + \frac{\gamma}{h}\right) - \frac{4}{6h}\gamma p_i - \frac{q_{i-1}}{6}$$

$$F_i = \frac{4p_i}{6}\left(1 - \frac{\gamma}{h}\right) + \frac{4}{6}q_i - \frac{\gamma p_{i-1}}{6h} + \frac{\gamma p_{i+1}}{6h}$$

$$G_i = \frac{1}{2h} - \frac{p_{i+1}}{6}\left(1 - \frac{\gamma}{h}\right) - \frac{q_{i+1}}{6}$$

$$H_i = \frac{\left(r_{i-1} + 4r_i + r_{i+1}\right)}{6}$$

The equation (6) gives tridiagonal system of (N-1) equations with (N+1) unknowns y_0 to y_N. The two given boundary conditions (2) together with these (N-1) equations are then sufficient to solve for the unknowns y_i's by using an efficient algorithm called discrete invariant imbedding.

2.2 Right –End Boundary Layer

Now assume that $b(x) \le 0$, $a(x) \le M < 0$ throughout the interval [0, 1], where M is some negative constant. Under these assumptions, (1) - (2) has a unique solution $y(x)$ which in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 1$ for small values of ε.

Let γ be a small positive deviating argument, $0 < \gamma << 1$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y\left(x + \gamma\right) = y(x) + \gamma y'(x) + \frac{\gamma^2}{2}y''(x)$$

$$y''(x) = \frac{2y\left(x + \gamma\right) - 2y(x) - 2\gamma y'(x)}{\gamma^2} \tag{7}$$

and consequently, equation (1) is replaced by the following first order differential equation with a small deviation argument:

$$y'(x) = p(x)y(x+\gamma) + q(x)y(x) + r(x)\text{, for } 0 \le x \le 1-\gamma \tag{8}$$

where

$$p(x) = \frac{-2\varepsilon}{-2\gamma\varepsilon + \gamma^2 a(x)},\ q(x) = \frac{2\varepsilon - \gamma^2 b(x)}{-2\gamma\varepsilon + \gamma^2 a(x)}$$

$$r(x) = \frac{\gamma^2 f(x)}{-2\gamma\varepsilon + \gamma^2 a(x)}$$

Taking fourth order finite difference approximation $y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h\left(1 + \frac{\delta^2}{6}\right)} + O\left(h^4\right)$ in the above equation, we get

$$\frac{y_{i+1} - y_{i-1}}{2h\left(1 + \frac{\delta^2}{6}\right)} = p(x_i)y(x_i + \delta) + q(x_i)y(x_i) + r(x_i)$$

$$\frac{y_{i+1} - y_{i-1}}{2h} = \left(1 + \frac{\delta^2}{6}\right)\left(p(x_i)y(x_i + \delta) + q(x_i)y(x_i) + r(x_i)\right)$$

After simple manipulation, we get three term recurrence relation given by

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i,\quad \text{for } i = 1, 2, \ldots, N\text{-1} \tag{9}$$

where

$$E_i = \frac{-1}{2h} - \frac{p_{i-1}}{6}\left(1-\frac{\gamma}{h}\right) - \frac{q_{i-1}}{6}$$

$$F_i = \frac{4p_i}{6}\left(1-\frac{\gamma}{h}\right) + \frac{4}{6}q_i - \frac{\gamma p_{i+1}}{6h} + \frac{\gamma p_{i-1}}{6h}$$

$$G_i = \frac{1}{2h} - \frac{p_{i+1}}{6}\left(1+\frac{\gamma}{h}\right) - \frac{q_{i+1}}{6} - \frac{4}{6h}p_i\gamma$$

$$H_i = \frac{(r_{i-1}+4r_i+r_{i+1})}{6}$$

The equation (9) gives tridiagonal system and we can solve this system together with the given boundary conditions using discrete invariant imbedding algorithm.

3 Numerical Examples

To demonstrate the applicability of the method we have applied it on three linear and two nonlinear singular perturbation problems with left-end boundary layer, two with right–end boundary layer problem. These examples have been chosen because they have been widely discussed in literature and because exact solutions are available for comparison.

Example 1: Consider the following homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) - y(x) = 0\,; \quad x \in [0,1]$$

with $y(0)=1$ and $y(1)=1$.

The exact solution is given by

$$y(x) = \frac{[(e^{m_2}-1)e^{m_1 x} + (1-e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]}$$

where $m_1=(-1+\sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2=(-1-\sqrt{1+4\varepsilon})/(2\varepsilon)$

The numerical results are given in table 1 for $\varepsilon=10^{-3}$, $h=10^{-2}$.

Example 2: Now consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1+2x\,; \quad x \in [0,1]$$

with $y(0)=0$ and $y(1)=1$.

The exact solution is given by

$$\mathrm{y(x) = x(x+1-2}\varepsilon) + \frac{(2\varepsilon-1)\left(1-e^{-x/\varepsilon}\right)}{\left(1-e^{-1/\varepsilon}\right)}$$

The numerical results are given in table 2 for $\varepsilon=10^{-3}$, $h=10^{-2}$.

Example 3: Consider the following variable coefficient singular perturbation problem

$$\varepsilon y''(x) + (1-\frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0\,; \quad x \in [0,1]$$

with $y(0)=0$ and $y(1)=1$.

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [7] page 148; equation 4.2.32] as our 'exact' solution:

$$y(x) = \frac{1}{2-x} - \frac{1}{2}e^{-(x-x^2/4)/\varepsilon}$$

The numerical results are given in table 3 for $\varepsilon=10^{-3}$, $h=10^{-2}$.

Example 4: Consider the following non linear singular perturbation problem

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0\,; \quad x \in [0,1]$$

with $y(0)=0$ and $y(1)=0$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1}y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

We have chosen to use Bender and Orszag's uniformly valid approximation [1], page 463; equation: 9.7.6] for comparison,

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)\,e^{-2x/\varepsilon}$$

For this example, we have boundary layer of thickness O(ε) at x=0. [cf. Bender and Orszag [1]].

The numerical results are given in table 4 for $\varepsilon=10^{-3}$, $h=10^{-2}$.

Example 5: Consider the following nonlinear singular perturbation problem from O'Malley [8], page 9; equation (1.10) case 2]:

$$\varepsilon y''(x) - y(x)y'(x) = 0\,; \quad x \in [-1,1]$$

with $y(-1)=0$ and $y(1)=-1$.

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0$$

We have chosen to use O' Malley's approximate solution [[8], pages 9 and 10; equations 1.13 and 1.14] for comparison,

$$y(x) = -\frac{\left(1-e^{-(x+1)/\varepsilon}\right)}{\left(1+e^{-(x+1)/\varepsilon}\right)}$$

For this example, we have a boundary layer of width O(ε) at x= -1 [cf. O' Malley [8], pages 9 and 10, eqs. (1.10), (1.13), (1.14), case 2] .

The numerical results are given in table 5 for $\varepsilon=10^{-3}$, $h=10^{-2}$.

Example 6: Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0\,; \quad x \in [0,1]$$

with $y(0)=1$ and $y(1)=0$.

Clearly, this problem has a boundary layer at x=1 i.e., at the right end of the underlying interval.

The exact solution is given by

$$y(x) = \frac{\left(e^{(x-1)/\varepsilon}-1\right)}{\left(e^{-1/\varepsilon}-1\right)}.$$

The numerical results are given in table 6 for $\varepsilon=10^{-3}$, $h=10^{-2}$.

Example7: Now we consider the following singular perturbation Problem

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0\,; x\in[0,1]$$

with $y(0)=1+exp(-(1+\varepsilon)/\varepsilon)$; and $y(1)=1+1/e$.

Clearly this problem has a boundary layer at x=1.

The exact solution is given by

$$y(x)=e^{(1+\varepsilon)(x-1)/\varepsilon}+e^{-x}$$

The numerical results are given in table 7 for $\varepsilon=10^{-3}$, $h=10^{-2}$.

4 Discussions and Conclusions

In this paper, we have developed a finite difference method to solve singularly perturbed two-point boundary value problem using deviating argument. We have replaced the given second order boundary value problem by an asymptotically equivalent first order differential equation with deviating argument. We have applied a fourth order finite difference approximation for the first derivative and obtained a tridiagonal system. Then, we have solved this tridiagonal system efficiently by discrete invariant imbedding algorithm. This method is iterative on deviating argument. We have presented numerical results of several model linear and nonlinear examples for different value of deviating argument to support the proposed method.

Table 1

Numerical results of Example 1 with $h=10^{-2},\ \varepsilon=10^{-3}$

X	y with $\gamma=\varepsilon$	y with $\gamma=1.2\varepsilon$	y with $\gamma=1.3\varepsilon$	Exact solution
0	1	1	1	1
0.01	0.462307	0.405503	0.380332	0.371972
0.02	0.390121	0.378952	0.377257	0.375678
0.03	0.383246	0.381190	0.380955	0.379450
0.04	0.385562	0.384922	0.384763	0.383259
0.05	0.389199	0.388765	0.388610	0.387107
0.1	0.408984	0.408585	0.408431	0.406935
0.3	0.498877	0.498498	0.498352	0.496932
0.5	0.608529	0.608199	0.608072	0.606833
0.7	0.742281	0.742040	0.741947	0.741040
0.9	0.905432	0.905334	0.905296	0.904927
1	1	1	1	1

Max.error: 0.0903 0.0335 0.0084

Table 2

Numerical results of Example 2 with $h=10^{-2},\ \varepsilon=10^{-3}$

x	y with $\gamma=\varepsilon$	y with $\gamma=1.2\varepsilon$	y with $\gamma=1.3\varepsilon$	Exact solution
0	0	0	0	0
0.01	-0.883171	-0.926853	-0.967217	-0.987874
0.02	-0.958637	-0.965971	-0.969390	-0.977639
0.03	-0.956471	-0.958178	-0.959149	-0.967159
0.04	-0.946680	-0.947725	-0.948553	-0.956480
0.05	-0.935976	-0.936944	-0.937756	-0.945600
0.1	-0.879090	-0.879999	-0.880769	-0.888200
0.3	-0.601515	-0.602222	-0.602820	-0.608600
0.5	-0.243939	-0.244444	-0.244871	-0.249000
0.7	0.1936363	0.1933333	0.1930769	0.1906000
0.9	0.7112121	0.7111111	0.7110256	0.7102000
1	1	1	1	1

Max.error: 0.1047 0.0610 0.0207

Table 3

Numerical results of Example 3 with $h=10^{-2},\ \varepsilon=10^{-3}$

x	y with $\gamma=\varepsilon$	y with $\gamma=1.2\varepsilon$	y with $\gamma=1.3\varepsilon$	Exact solution
0	0	0	0	0
0.01	0.457981	0.479457	0.499300	0.502489
0.02	0.504599	0.507586	0.508695	0.505050
0.03	0.511543	0.511592	0.511376	0.507614
0.04	0.514585	0.514272	0.513973	0.510204
0.05	0.517254	0.516900	0.516595	0.512820
0.1	0.530782	0.530421	0.530114	0.526315
0.3	0.592740	0.592377	0.592068	0.588235
0.5	0.670989	0.670641	0.670346	0.666666
0.7	0.772878	0.772586	0.772337	0.769230
0.9	0.910908	0.910764	0.910641	0.909090
1	1	1	1	1

Max.error: 0.0445 0.0230 0.0038

TABLE 4

NUMERICAL RESULT OF EXAMPLE 4 WITH $h = 10^{-2},\ \varepsilon = 10^{-3}$

X	y with $\gamma = 0.5\varepsilon$	y with $\gamma = 0.6\varepsilon$	y with $\gamma = 0.7\varepsilon$	Exact solution
0	0	0	0	0
0.01	0.586235	0.648906	0.702389	0.683196
0.02	0.663050	0.674985	0.676164	0.673344
0.03	0.665811	0.667107	0.666704	0.663588
0.04	0.657880	0.657465	0.656958	0.653926
0.05	0.648475	0.647816	0.647316	0.644357
0.10	0.601522	0.600899	0.600456	0.597837
0.30	0.433024	0.432646	0.432377	0.430782
0.50	0.288977	0.288759	0.288603	0.287682
0.70	0.163163	0.163055	0.162977	0.162518
0.90	0.051475	0.051444	0.051422	0.051293
1	0	0	0	0

Max.error: 0.0970 0.0343 0.0192

TABLE 5

NUMERICAL RESULT OF EXAMPLE 5 WITH $h = 10^{-2},\ \varepsilon = 10^{-3}$

X	y with $\gamma = 1.1\varepsilon$	y with $\gamma = 1.2\varepsilon$	y with $\gamma = 1.3\varepsilon$	Exact solution
-1	0	0	0	0
-0.99	-0.904109	-0.947368	-0.987341	-0.999909
-0.98	-0.990805	-0.997229	-0.999839	-0.999999
-0.97	-0.999118	-0.999854	-0.999997	-0.999999
-0.96	-0.999915	-0.999992	-0.999999	-1.000000
-0.95	-0.999991	-0.999999	-0.999999	-1.000000
-0.70	-1.000000	-1.000000	-1.000000	-1.000000
-0.50	-1.000000	-1.000000	-1.000000	-1.000000
-0.30	-1.000000	-1.000000	-1.000000	-1.000000
0.10	-1.000000	-1.000000	-1.000000	-1.000000
0.30	-1.000000	-1.000000	-1.000000	-1.000000
0.50	-1.000000	-1.000000	-1.000000	-1.000000
0.70	-1.000000	-1.000000	-1.000000	-1.000000
0.90	-1.000000	-1.000000	-1.000000	-1.000000
1	-1.000000	-1.000000	-1.000000	-1.000000

Max.error: 0.0958 0.0525 0.0126

TABLE 6

NUMERICAL RESULT OF EXAMPLE 6 WITH $h = 10^{-2},\ \varepsilon = 10^{-3}$

x	y with $\gamma = 1.1\varepsilon$	y with $\gamma = 1.2\varepsilon$	y with $\gamma = 1.3\varepsilon$	Exact solution
0	1	1	1	1.000000
0.10	1.000000	0.999999	0.999999	1.000000
0.30	1.000000	0.999999	0.999999	1.000000
0.50	1.000000	0.999999	0.999999	1.000000
0.70	1.000000	0.999999	0.999999	1.000000
0.90	0.999999	0.999999	0.999999	1.000000
0.95	0.999991	0.999999	0.999999	1.000000
0.96	0.999915	0.999992	0.999999	1.000000
0.97	0.999118	0.999854	0.999997	0.999999
0.98	0.990805	0.997229	0.999839	0.999999
0.99	0.904109	0.947368	0.987341	0.999954
1	0	0	0	0

Max. error: 0.0958 0.0526 0.0126

TABLE 7

NUMERICAL RESULT OF EXAMPLE 7 WITH $h = 10^{-2},\ \varepsilon = 10^{-3}$

x	y with $\gamma = 1.1\varepsilon$	y with $\gamma = 1.2\varepsilon$	y with $\gamma = 1.3\varepsilon$	Exact solution
0	1	1	1	1
0.10	0.905289	0.905245	0.905207	0.904837
0.30	0.741930	0.741820	0.741726	0.740818
0.50	0.608048	0.607898	0.607771	0.606530
0.70	0.498326	0.498153	0.498007	0.496585
0.90	0.408403	0.408221	0.408067	0.406569
0.95	0.388589	0.388400	0.388245	0.386741
0.96	0.384811	0.384559	0.384397	0.382892
0.97	0.381744	0.380872	0.380589	0.379083
0.98	0.385897	0.379530	0.376936	0.375311
0.99	0.466836	0.423779	0.383986	0.371621
1	1.367879	1.367879	1.367879	1.367879

Max.error: 0.0952 0.0522 0.0124

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